

RIGIDITY OF MULTIPARAMETER ACTIONS

BY

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ABSTRACT

We survey some of the recent progress in understanding diagonalizable algebraic actions of multidimensional abelian groups, a subject pioneered by Hillel Furstenberg.

0. Prologue

In this survey I would like to expose recent developments and applications of the study of the rigidity properties of natural algebraic actions of multidimensional abelian groups. This study was initiated by Hillel Furstenberg in his landmark paper [Fur67].

The survey is an outgrowth of notes I prepared for a series of lectures in the II Workshop on Dynamics and Randomness in December 2002; a related talk was given also in the Conference on Probability in Mathematics in honor of Hillel Furstenberg. It is not intended to be an exhaustive survey, and is skewed towards the problems I have worked on and which I know best. In particular, I make no mention of \mathbb{Z}^d -actions on zero-dimensional compact groups such as Ledrappier's example of a 2-mixing but not 3-mixing \mathbb{Z}^2 -action [Sch95, Intr.] — a deficiency that is partially mitigated by the existence of several good surveys, for example [Sch03].

It has been my good fortune to have been able to learn ergodic theory from Benjamin Weiss, Hillel's long-term collaborator and friend, and from Hillel himself during my graduate studies in The Hebrew University. I was introduced to

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the problems mentioned here by numerous discussions (as well as collaborations) with David Meiri, Shahar Mozes, Yuval Peres and Barak Weiss, all of whom were students of Hillel, as well as with Hillel himself. It is therefore with great pleasure that I dedicate this survey to Hillel on the occasion of his retirement. After graduating, I met Peter Sarnak and learned from him many things, including about the quantum unique ergodicity problem, which is fortuitously connected to precisely these kind of questions. Without Peter's help and encouragement, at times even his insistence, I would not have been able to discover this connection to quantum unique ergodicity which motivated much of my work on this subject.

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1. Rigidity on the one-torus

One of the simplest dynamical systems is the map

$$\times_n: x \mapsto nx \bmod 1$$

on the unit interval, where n is any natural number. In order to make this map continuous, we think of it as a map on the 1-torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This map is essentially the same as the shift on a one-sided infinite sequence of n symbols: the exact relationship between these two systems is that there is a continuous surjection from the one-sided shift on n symbol to \mathbb{T} commuting with the respective \mathbb{Z}^+ -actions, and this continuous surjection is finite-to-1 everywhere (in fact at most 2-to-1) and is 1-to-1 except at countably many points.

There are very many closed invariant sets for the map \times_n , and very many invariant or even ergodic measures. As discovered by Furstenberg, the situation changes dramatically if one adds an additional map into the picture. One may consider the two commuting maps $\times_n: x \mapsto nx \bmod 1$ and $\times_m: x \mapsto mx \bmod 1$; in order to really have two different maps it is natural to assume that n and m are **multiplicative independent**, i.e., that $\log n / \log m \notin \mathbb{Q}$.

In this case Furstenberg proved that there are very few closed invariant sets [Fur67], and conjectured that a similar statement holds for invariant measures.

THEOREM 1.1 (Furstenberg [Fur67]): *Let n, m be two multiplicative independent positive integers. Assume that $X \subset \mathbb{T}$ is closed and invariant under \times_n and \times_m (i.e., that $\times_n(X) \subset X$ and similarly for \times_m). Then either $X = \mathbb{T}$ or X is finite (and so necessarily $X \subset \mathbb{Q}$).*

His proof is a true gem and uses a fundamental idea, which was new at the time, namely disjointness of dynamical systems: some systems are so different they cannot be coupled in any nontrivial way. Like most of what we consider here, this notion applies both in the measurable category and in the topological category; here is the topological version:

Definition 1.2: Let G be a reasonable semigroup (say locally compact, second countable). Suppose G acts continuously on two spaces X and Y . A **joining** of (X, G) and (Y, G) is a closed subset $Z \subset X \times Y$ invariant under the diagonal G action on $X \times Y$ whose projection to the first coordinate is X and to the second coordinate is Y . The two systems (X, G) and (Y, G) are said to be **disjoint** if the only joining between them is the trivial product joining $Z = X \times Y$.

In Furstenberg's proof of Theorem 1.1 a key point is that a continuous \mathbb{Z}^d -action on a compact space with many periodic points is disjoint from any minimal \mathbb{Z}^d -action (i.e., one in which every orbit is dense).

We now present Furstenberg's conjecture about measure rigidity, which despite substantial progress is still open and considered one of the most outstanding problems in ergodic theory. We recall that a measure μ on a measurable space (X, \mathcal{B}) on which a semigroup G acts is said to be **invariant** if for every $g \in G$ and any $A \in \mathcal{B}$ we have that $\mu(g^{-1}A) = \mu(A)$; an invariant measure is said to be **ergodic** if every set $A \in \mathcal{B}$ which is invariant under every $g \in G$ satisfies that $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

CONJECTURE 1 (Furstenberg (1967)): *Let n, m be two multiplicative independent positive integers. Any Borel measure μ on \mathbb{T} ergodic under the action of the semigroup generated by \times_n, \times_m is either Lebesgue measure or an atomic measure supported on finitely many rational points.*

generalizations are the main topic of these lectures. The first substantial partial result in this direction is due to R. Lyons [Lyo88]. He proved the following:

THEOREM 1.3 (Lyons [Lyo88]): *Let m, n be two multiplicative independent positive integers, and μ a \times_n, \times_m -ergodic measure on \mathbb{T} as in Furstenberg's conjecture. Assume further that with respect to the \mathbb{Z}^+ -action generated by the \times_n map, μ has the K property (i.e., $(\mathbb{T}, \mathcal{B}, \mu, \times_n)$ has no zero entropy factors). Then μ is Lebesgue measure.*

This theorem has been substantially strengthened by Rudolph, whose theorem (at least for the case of m, n relatively prime) still presents the state-of-the-art regarding Furstenberg's conjecture:

THEOREM 1.4 (Rudolph [Rud90]): *Let m, n be relatively prime positive integers, and μ a \times_n, \times_m -ergodic measure on \mathbb{T} as before. Assume that the entropy of the \mathbb{Z}^+ -action generated by one of these maps, say \times_n , has positive entropy. Then μ is Lebesgue measure.*

Rudolph's proof used in an essential way that n and m are relatively prime. This restriction was lifted by A. Johnson [Joh92] who extended Theorem 1.4 to n and m multiplicatively independent.

In his proof, Rudolph identified and dealt with a substantial complication, namely that even though μ is \times_n, \times_m -ergodic it does not need to be \times_n -ergodic. Overcoming this requires a new idea, and if one makes this ergodicity assumption, Rudolph's proof can be simplified. Note that by definition this difficulty does not exist if the action of \times_n on $(\mathbb{T}, \mathcal{B}, \mu)$ is K, hence in particular ergodic, which is a case covered by Lyons' theorem.

J. Feldman has been able to give an alternative proof of Rudolph's and Johnson's theorems using a method similar to that of Lyons [Fel93]; however, this proof does not seem very suitable for generalizations. A third approach to Rudolph's Theorem by Host [Hos95, Hos00] will play a key role later on.

An important advance was made by Katok and Spatzier [KS96, KS98], who discovered that Rudolph's proof can be extended to give partial information on invariant measures in much greater generality. Here, overcoming the need for an ergodicity assumption becomes substantially harder; an extension of Rudolph's original method of overcoming this difficulty works only in a rather restricted class of examples (of course, much more general than that considered in Rudolph's original proof) which satisfy a condition Katok and Spatzier called **total non-symplecticity (TNS)**.

In this survey we emphasize recent progress in the study of measures invariant under algebraic multiparameter Abelian actions, in particular progress towards the classifying of invariant measures without need for an ergodicity assumption in a much wider class of actions than the class of TNS actions for which this was previously known. It turns out that in applications to number theoretic and other problems this improvement is crucial. Also, following Host's approach mentioned above the scope of these techniques has been extended to deal with certain "partially isometric" **one** parameter actions.

As we will explain in §6, this study of invariant measure yields a proof of arithmetic cases of the quantum unique ergodicity conjecture [Lin03a]. We also mention that in [EKL04] (which is joint with M. Einsiedler and A. Katok) we use a partial classification of invariant measures to derive a substantial partial result

towards the long-standing Littlewood Conjecture on simultaneous Diophantine approximations (Theorem 3.1 below). In both cases measure rigidity techniques prove results which are at present unapproachable by other methods.

2. Rigidity for toral automorphism

In this and the next section, we present two classes of actions of multidimensional abelian groups where similar rigidity phenomena can be found.

Much of what has been said in the first section regarding the \times_m map is also true for higher-dimensional toral automorphisms, or more generally for automorphisms of connected abelian groups. If α is a single toral automorphism, and assuming the automorphism is expansive (i.e., hyperbolic), one can find a sofic system (i.e., a factor of a shift of finite type) which is related to the toral automorphism in a way which is very similar to how the one-sided shift on n symbols is related to the \times_n map, and so the abundance of closed invariant sets and invariant measures on a sofic system implies that the same holds for a hyperbolic toral automorphism.

The construction of such sofic models is a fascinating story in its own right. A geometric construction in the two-dimensional case was discovered by Adler and Weiss [AW67], and this theory, the theory of Markov partitions, was developed by Sinai, Bowen and others as an exceedingly powerful tool for the study of general Anosov and axiom A systems. More recently, Kenyon and Vershik [KV98] (following earlier work of Vershik) constructed an algebraic coding specifically for hyperbolic toral automorphisms which better reflects the algebraic structure of this system; a related but different coding has been developed by K. Schmidt [Sch00], and more careful analysis of certain special cases was carried out by Sidorov and Vershik [SV98].

In general, automorphisms of a multidimensional torus fail to commute and so typically a finite set of automorphisms generates a rather large group. An extreme case is the study of actions of groups of automorphisms which are commensurable to an arithmetic lattice in a higher-rank Lie group. Certainly in this case one expects very few closed invariant sets and invariant measures, but much of the rigidity comes from the group itself and not from the particular action at hand. The study of actions of these “big” groups is a deep and fascinating topic in its own right, but has a completely different flavor and is beyond the scope of this paper.

Here we focus on the other extreme, the study of measures and closed sets in \mathbb{T}^d invariant under a multidimensional (i.e., not virtually cyclic) Abelian group

of toral automorphisms. Certainly there is nothing rigid about the way \mathbb{Z}^n acts per se: virtually anything which is known about \mathbb{Z} -actions is also known for \mathbb{Z}^n -actions: indeed for actions of amenable groups [OW87]. The surprising fact is that many natural \mathbb{Z}^n - or \mathbb{R}^n -actions have (at least conjecturally) this unexpected scarcity of closed invariant sets and invariant measures.

Some care is needed when stating the exact conditions for rigidity. For example, consider the \mathbb{Z}^2 -action α generated by the two commuting toral automorphisms on \mathbb{T}^4 given by the integer matrices

$$(2.1) \quad A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This action is a faithful \mathbb{Z}^2 -action (i.e., no $\mathbf{n} \in \mathbb{Z}^2$ acts trivially) but it is clear that no measure or topological rigidity can hold here since the projection $\pi: \mathbb{T}^4 \rightarrow \mathbb{T}^2$ obtained by omitting the two last coordinates is a factor map that intertwines the \mathbb{Z}^2 -action α on \mathbb{T}^4 with a \mathbb{Z} -action on \mathbb{T}^2 . There is no shortage of closed invariant sets and invariant measures for any \mathbb{Z} -action by a hyperbolic toral automorphism on \mathbb{T}^2 , and any such invariant set or invariant measure can be lifted to an invariant set or measure respectively on (\mathbb{T}^4, α) . It is a conjecture of Katok and Spatzier that this type of construction is the only source of non-algebraic α ergodic measures for a \mathbb{Z}^d -action on a torus with $d > 1$.

A \mathbb{Z}^d -action α on \mathbb{T}^n is said to be **irreducible** if the only closed, α -invariant proper subgroups of \mathbb{T}^n are finite. Such actions do not have any lower-dimensional factors (since otherwise the kernel of the factor map will be a closed infinite proper subgroup). It is **strongly irreducible** if the same holds for every finite index subgroup of \mathbb{Z}^d . For simplicity we assume that α is a faithful, strongly irreducible \mathbb{Z}^d -action on \mathbb{T}^n for some $d > 1$, and then the above method of creating strange invariant sets and measures is not applicable, though even if one understands well the irreducible case, the passage to the general case involves new issues (see, e.g., [KK02, KS96, KS98, Mei98, MP99]). We also mention that some of the most interesting applications have involved considering non-irreducible actions [KK02, KKS02].

Using the same basic strategy as Furstenberg, D. Berend extended Furstenberg's theorem to the context of semigroups of commuting endomorphisms of a torus and more general abelian groups [Ber84a, Ber84b, Ber8e].

As mentioned earlier, Katok and Spatzier generalized Rudolph's proof to this context (as well as to the type of examples considered in the next section).

As before, one needs an assumption about positive entropy of the \mathbb{Z} -action generated by $\alpha_{\mathbf{n}}$ for some $\mathbf{n} \in \mathbb{Z}^d$. But in most cases, an ergodicity assumption is also needed. We only discuss the case of irreducible actions, though Katok and Spatzier treat also the general case.

THEOREM 2.1 (Katok and Spatzier [KS96, KS98]): *Suppose α is a strongly irreducible \mathbb{Z}^d faithful action on \mathbb{T}^n with $d > 1$. Let μ be an α -invariant and ergodic measure on \mathbb{T}^n , and suppose that for some $\mathbf{n} \in \mathbb{Z}^d$, the entropy of μ with respect to the \mathbb{Z} -action generated by $\alpha_{\mathbf{n}}$ is positive. Suppose further that one of the following holds:*

- (1) *the \mathbb{Z}^d -action α on \mathbb{T}^n is totally non-symplectic (we do not provide the exact definition here but, for example, a faithful \mathbb{Z}^d -action on \mathbb{T}^{d+1} is always of this kind);*
- (1) *the action given by α on the measure space (X, \mathcal{B}, μ) satisfies some ergodicity conditions; a natural condition to use is that α is mixing as a \mathbb{Z}^d -action which in particular implies that for any $\mathbf{n} \in \mathbb{Z}^d$, we have that μ is ergodic for the \mathbb{Z} -action generated by $\alpha_{\mathbf{n}}$.*

Then μ is Lebesgue measure.

Jointly with M. Einsiedler, we have been able to eliminate the need for assuming an ergodicity condition or total non-symplecticity.

THEOREM 2.2 (Einsiedler and Lindenstrauss [EL03]): *Suppose α is a strongly irreducible \mathbb{Z}^d faithful action on \mathbb{T}^n with $d > 1$. Let μ be an α -invariant and ergodic measure on \mathbb{T}^n , and suppose that for some $\mathbf{n} \in \mathbb{Z}^d$, the entropy of μ with respect to the \mathbb{Z} -action generated by $\alpha_{\mathbf{n}}$ is positive. Then μ is Lebesgue measure.*

We remark that the techniques of [EL03] apply equally well to the reducible case, but necessarily the statement in this case is more cumbersome. We refer to [EL03] for additional details.

The proof of Theorem 2.2 uses a new observation regarding measures invariant under a **single**, possibly hyperbolic, toral automorphism. In order to present this observation, we first need to explain how one can attach for any sub-foliation of the unstable foliation its contribution to the entropy.

Let $T: x \mapsto Ax \bmod \mathbb{Z}^n$ be such an irreducible toral automorphism, with $A \in \mathrm{GL}(n, \mathbb{Z})$, and μ a T -invariant measure. Let Φ^+ be the set of eigenvalues of A with absolute value > 1 . To any nonempty $S \subset \Phi^+$ which is closed under complex conjugation we set $W_S \subset \mathbb{R}^n$ to be the space spanned by the

corresponding eigenvectors, and the orbits $x + W_S$ of this space form a T -invariant expanding foliation of \mathbb{T}^n . In this case one can define an entropy contribution $h_\mu(T, W_S)$ which roughly corresponds to the part of the total metric (i.e., measure theoretic) entropy $h_\mu(T)$ of T which can be detected using the conditional measure μ induces on W_S -leaves, and these satisfy the following properties:

- (1) for any S as above, $h_\mu(T) \geq h_\mu(T, W_S)$, with equality for $S = \Phi^+$.
- (1) $0 \leq h_\mu(T, W_S) \leq \sum_{s \in S} \log |s|$; for Lebesgue measure λ , $h_\lambda(T, W_S) = \sum_{s \in S} \log |s|$.

As is often the case in ergodic theory, one is also interested in the corresponding relative quantities: if \mathcal{A} is a T -invariant sigma algebra of Borel subsets of \mathbb{T}^n , one can define the conditional entropies $h_\mu(T|\mathcal{A})$ and $h_\mu(T, W_S|\mathcal{A})$ which have similar properties. Precise definitions of all these quantities and additional references can be found in [EL03, Sect. 3].

The basic inequality at the heart of the approach of [EL03] is the following:

THEOREM 2.3 ([EL03, Thm. 4.1]): *Let T be an irreducible toral automorphism, and μ a T -invariant measure. Let $W_S \subset \mathbb{R}^n$ be an expanding T -invariant subspace as above. Then*

$$(2.2) \quad h_\mu(T, W_S) \leq \frac{h_\lambda(T, W_S)}{h_\lambda(T)} h_\mu(T).$$

More generally, for every T -invariant sigma algebra \mathcal{A} we have

$$h_\mu(T, W_S|\mathcal{A}) \leq \frac{h_\lambda(T, W_S)}{h_\lambda(T)} h_\mu(T|\mathcal{A}).$$

We emphasize that (2.2) is false for reducible T . It is an interesting question when equality can be attained.

Returning to the case of an irreducible \mathbb{Z}^d -action α for $d > 1$, in conjunction with a lemma from [EK03], Theorem 2.3 implies that for any α -invariant μ and \mathcal{A} there is a single constant $c_{\mu, \mathcal{A}} \in [0, 1]$ such that for any $\mathbf{n} \in \mathbb{Z}^d$

$$(2.3) \quad h_\mu(\alpha_{\mathbf{n}}|\mathcal{A}) = c_{\mu, \mathcal{A}} h_\lambda(\alpha_{\mathbf{n}}).$$

This one-dimensionality of the entropy function for α -invariant measures can be expected in the one-dimensional case covered by Rudolph's theorem (Theorem 1.4), where entropy is closely related to Hausdorff dimension of μ , and indeed (2.3) plays a prominent role in Rudolph's proof (see [Rud90, Thm. 3.7]).

That it holds also in the higher-dimensional irreducible case is somewhat surprising, and allows one to prove Theorem 2.2 along the lines of Rudolph's original proof.¹

Finally, we mention that there has been some work lately on classifying closed subsets of \mathbb{T}^n invariant under an action by more general semigroups of automorphisms (or endomorphisms) of the torus [Muc00, Sta99, Sta00, Muc02]. All of these results assume in particular that the action is strongly irreducible and the group acting is not virtually cyclic; under these assumptions [Muc02] provides a complete classification.

3. Rigidity on homogeneous spaces

Here we present another class of examples where topological and measure rigidity of multiparameter actions is expected to hold.

A concrete and very interesting special case is the following: let

$$X = \mathrm{SL}(3, \mathbb{Z}) \backslash \mathrm{SL}(3, \mathbb{R}).$$

This is simply the space of lattices in \mathbb{R}^3 of covolume one. It is not a compact space: if one takes a sequence of lattices Λ_i in \mathbb{R}^3 with each Λ_i containing a vector v_i with $\|v_i\| \rightarrow 0$, then no subsequence of Λ_i has a limit in X . However, X has a (unique) probability measure which is invariant under the natural action of $\mathrm{SL}(3, \mathbb{R})$ on X . The diagonal matrices

$$H = \left\{ \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^{-s-t} \end{pmatrix} : s, t \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R})$$

give a very nice \mathbb{R}^2 action on X by multiplication from the right.

We mention in passing that for any one-parameter subgroup of H there is a big collection of closed invariant sets and of invariant measures; but these collections are much less understood than in the case of (hyperbolic) toral automorphisms. We also mention that there are known irregular H -invariant closed subsets of X arising, as in the previous section, from special situations in which the action is essentially reduced to an action of a one-parameter group. For this particular X , this can happen if we are considering the orbit closure of a point $\Lambda \in X$ under the

¹ The alert reader may notice that in Theorem 2.2 strong irreducibility is assumed while in Theorem 2.3 no such assumption is made. The reason is that in Rudolph's proof strong irreducibility is implicitly assumed; indeed, Theorem 2.2 as stated is false in the irreducible but not strongly irreducible case.

H -action when there is one one-parameter subgroup of H , say $\{a(t) : t \in \mathbb{R}\}$, so that $a(t)\Lambda \rightarrow \infty$ as $|t| \rightarrow \infty$. For more general examples of this kind, such reduction to an action of a one-parameter group can happen even in the compact case. Note that when classifying invariant probability measures, by Poincaré recurrence, the particular situation described above cannot happen, though in general, the same kind of difficulties exist when classifying invariant probability measures (this complication has been pointed out by M. Rees in the unpublished [Ree82] and independently by Mozes in [Moz95]; a nice account of the Rees example and some generalizations can be found in [EK03, Section 9]).

To avoid this complication of possible bad behavior of certain diverging orbits, we can consider the following partial classification of H orbits on $X = \mathrm{SL}(3, \mathbb{Z}) \backslash \mathrm{SL}(3, \mathbb{R})$. We note that even for this specific X , this is only a special case of a more general conjecture regarding closed H -orbits.

CONJECTURE 2 (G. Margulis [Mar97]): *Any compact H -invariant subset of $\mathrm{SL}(3, \mathbb{Z}) \backslash \mathrm{SL}(3, \mathbb{R})$ is a union of compact H -orbits.*

It turns out that even this very special case of topological rigidity for homogeneous spaces implies the following long-standing conjecture of Littlewood:

CONJECTURE 3 (Littlewood (c. 1930)): *Let $\|x\|$ denote the distance from $x \in \mathbb{R}$ to the closest integer. Then*

$$(3.1) \quad \lim_{n \rightarrow \infty} n \|n\alpha\| \|n\beta\| = 0$$

for any real numbers α and β .

This implication was discovered in a different terminology, long before Furstenberg's pioneering work regarding the rigidity of multiparameter actions, by Cassels and Swinnerton-Dyer [CSD55]; however, it was Margulis who first recast this in dynamical terms [Mar97]. We note that the current state-of-the-art regarding Conjecture 2, Theorem 3.4 below, while not sufficient to imply Littlewood's conjecture, does imply the following nontrivial estimate towards this conjecture:

THEOREM 3.1 (Einsiedler, Katok and Lindenstrauss [EKL04]): *The set of $(\alpha, \beta) \in \mathbb{R}^2$ for which (3.1) is not satisfied has Hausdorff dimension zero.*

A weaker result, namely that this Hausdorff dimension is at most one, can be derived from Theorem 3.2.

The most explicit form of a general rigidity conjecture for \mathbb{R}^d and more general actions on homogeneous spaces has been given by Margulis [Mar00]. A less

explicit conjecture in the same spirit has been given by Katok and Spatzier in [KS96].

We take G to be a connected Lie group, Γ a lattice in G , i.e., discrete subgroup of G of finite covolume, and $H < G$ a connected subgroup. One of the highlights of the theory of flows on homogeneous spaces is the classification of closed invariant sets and invariant measures on $\Gamma \backslash G$ when H is generated by unipotent one-parameter subgroups by M. Ratner [R91a, R91b]; some years earlier an important special case allowed Margulis to resolve the long-standing Oppenheim conjecture ([Mar89]; see [DM90] for an elementary treatment as well as the survey [Mar97] for more details); we also mention that this measure and topological rigidity of unipotent flows was conjectured by Dani and Raghunathan respectively. A valuable recent addition to the literature on this deep and interesting subject is [Mor03].²

Taking H to be as in Conjecture 2 is perhaps a prototypical example of the kind of groups not covered by Ratner's theorem. In his conjecture, Margulis only assumes that H is generated by elements $h \in H$ so that all eigenvalues of $Ad_G(h)$ are real. Margulis gives a conjecture both in the topological category and in the measure theoretic category:

CONJECTURE 4 (Margulis [Mar00]): *Let G , Γ and H be as above. For any $x \in \Gamma \backslash G$ one of the following holds:*

- (1) \overline{Hx} is homogeneous, i.e., is a closed orbit of a closed connected subgroup $L < G$;
- (2) *there exists a closed connected subgroup $F < G$ and a continuous epimorphism ϕ of F onto a Lie group L such that $H < F$, Fx is closed in $\Gamma \backslash G$, $\phi(F_x)$ is closed in L where F_x denote the stabilizer $\{g \in F : gx = x\}$, and $\phi(H)$ is a one-parameter subgroup of L containing no nontrivial Ad_L -unipotent elements.*

CONJECTURE 5 (Margulis [Mar00]): *Let G , Γ and H be as above. Let μ be an H -invariant H -ergodic measure on $\Gamma \backslash G$. Suppose that for every $x \in \text{supp } \mu$ the statement (2) in the formulation of Conjecture 4 does not hold. Then μ is algebraic (i.e., the invariant measure on the closed orbit of some subgroup $L < G$ containing H).*

2 The simplest case is the horocycle flow on compact surfaces of constant negative curvature, for which minimality (i.e., nonexistence of closed invariant subsets) is classical and due to Hedlund [Hed39]; unique ergodicity of this flow, which was the first result regarding measure rigidity of unipotent flows, has been proved by Furstenberg in [Fur73].

It seems to me that the second conjecture is overly restrictive. For the record, we state the following modification of Conjecture 5:

CONJECTURE 6: *Let G, Γ and H be as above. Let μ be an H -invariant H -ergodic measure on $\Gamma \backslash G$. Suppose that for μ -almost every x the statement (2) in the formulation of Conjecture 4 does not hold. Then μ is algebraic.*

Note that these conjectures give no information for H a one-parameter diagonalizable group; and despite the fact that they treat non-commutative H the most interesting case seems to be H commutative (see discussion and remarks following Conjecture 2 in [Mar00] as well as [KSS02, Sec. 4b]). From now on we will only consider actions of commutative H .

Unlike the case for the original \times_n, \times_m -problem and its extensions discussed in the previous section, the study of invariant measures seems to have fared better than that of the topological questions. Regarding the topological question, S. Mozes [Moz94] has proved the following: let $G = \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$, $H < G$ the product of the groups of diagonal matrices in the two factors of G , and Γ an irreducible lattice. Then if \overline{Hx} contains a compact H orbit, then \overline{Hx} is homogeneous. This result has a nice interpretation in terms of certain two-dimensional tilings. Jointly with Barak Weiss [LW01] we have proved a similar result for H the full diagonal group in $G = \mathrm{SL}(n, \mathbb{R})$ and Γ , e.g., $\mathrm{SL}(n, \mathbb{Z})$; a substantial complication is the existence of intermediate groups $H < L < G$ with closed orbits for n composite. Generalization to general G seems to require additional ideas. We also mention a recent paper by G. Tomanov and Barak Weiss [TW02] classifying closed H -orbits in $\Gamma \backslash G$ for algebraic groups and H a maximal \mathbb{R} -split torus (e.g., for $G = \mathrm{SL}(n, \mathbb{R})$, H is the group of positive diagonal matrices). Their treatment generalizes an unpublished result of Margulis (see the appendix to [TW02]). Finally, we remark that the latest results towards the measure theoretic conjecture have implications regarding orbit closures, for example regarding their possible Hausdorff dimensions (see [EKL04, Part 2] for more details).

We now turn to the measure theoretic conjecture. In their papers, Katok and Spatzier treat simultaneously the homogeneous case and the case of multidimensional action of toral automorphisms. However (at least when G is semi-simple and H is a multidimensional subgroup of the Cartan group of G), this action never satisfies their total non-symplecticity assumption. Because of this, in order to deduce that an H -invariant measure is homogeneous, both an ergodicity assumption and an assumption regarding entropy is needed.

The entropy assumption is a major disadvantage. While the Katok–Spatzier

method and its extensions has been used very effectively to prove isomorphism rigidity in many cases (i.e., that any measurable isomorphism between two multiparameter actions is algebraic; for a study of isomorphism rigidity in the locally homogeneous context we refer the reader to the recent [KS] by B. Kalinin and Spatzier), its applicability to number theoretic questions is extremely limited. The reason for this is that typically, when applying measure rigidity to prove e.g. equidistribution, one considers the weak* limit of a sequence of measures μ_i . While entropy is usually well behaved under weak* limits, mixing properties such as ergodicity are not. This principle is well illustrated in, e.g., the proof of Theorem 3.1 in [EKL04, Part 2] (see also [JR95] for a related discussion) and is implicit in [BL03, Lin03a].

Recently, two rather different and complimentary methods to overcome the need for an ergodicity assumption for the local homogeneous context have been discovered (yet another idea, which to date has not been used in this context but is very useful for, e.g., commuting automorphisms of tori, has been briefly exposed in the previous section). Chronologically, the first of these has been by Einsiedler and Katok [EK03]; another completely different approach was found shortly thereafter by the author [Lin03a]. This latter approach uses some ideas and techniques introduced by Ratner in her work on the rigidity of unipotent flows.

In their paper, Einsiedler and Katok deal with general \mathbb{R} -split groups G with rank > 1 ; for simplicity (and also because the result is strongest in this case) we specialize to the case of $G = \mathrm{SL}(n, \mathbb{R})$. In this case Einsiedler and Katok prove the following:

THEOREM 3.2 (Einsiedler and Katok [EK03]): *Let $G = \mathrm{SL}(n, \mathbb{R})$ and let $\Gamma < G$ be discrete. Let $H < G$ be the subgroup of positive diagonal matrices. Let μ be a H -invariant and ergodic measure on $\Gamma \backslash G$. Assume that the entropy of μ with respect to all one-parameter subgroups of H is positive. Then μ is the G -invariant measure on $\Gamma \backslash G$.*

The proof of this result uses heavily the non-commutativity of the one-parameter unipotent subgroups of G normalized by H ; such use is already hinted at in [KS96]. But it also employs a new observation regarding how a measure invariant under a multiparameter action decomposes when restricted to subfoliations of the stable foliation. This ingredient has already found several additional applications, including in [EL03].

Using the above-mentioned additional method, the following (and more general statements) can be proved (see the discussion following the closely related

Theorem 5.1). We note that the Einsiedler–Katok method gives very little in this case, and that the entropy assumption is weaker than that of Theorem 3.2.

THEOREM 3.3 (Lindenstrauss [Lin03a]): *Let $G_1 = G_2 = \mathrm{SL}(2, \mathbb{R})$, $G = G_1 \times G_2$, and Γ be an irreducible lattice in G (i.e., a lattice in G whose projections to G_1 and G_2 are dense). Let H be the product of the group of positive diagonal matrices in G_1 with the group of positive diagonal matrices in G_2 . Suppose μ is H invariant and ergodic, and suppose that the entropy of μ with respect to the action of the one-parameter group of positive diagonal matrices in G_1 is positive. Then μ is the G -invariant measure on $\Gamma \backslash G$.*

It turns out that the argument of [Lin03a] is applicable for $\Gamma \backslash \mathrm{SL}(n, \mathbb{R})$ precisely for positive entropy measures for which the argument of [EK03] does not apply(!); combining these arguments one proves the following strengthening³ of Theorem 3.2:

THEOREM 3.4 (Einsiedler, Katok and Lindenstrauss [EKL04]): *Let $G = \mathrm{SL}(n, \mathbb{R})$ and $H < G$ be the subgroup of positive diagonal matrices. Let μ be a H -invariant and ergodic measure on $\mathrm{SL}(n, \mathbb{Z}) \backslash G$. Assume that there is some one-parameter subgroup of H with respect to which μ has positive measure. Then μ is algebraic and is not compactly supported. If n is prime then μ is the G -invariant measure on $\mathrm{SL}(n, \mathbb{Z}) \backslash G$.*

4. Recurrent measures

Consider an action of a locally compact group M on a complete separable metric space X . We say that the measure μ on X is M -recurrent if for every subset $B \subset X$ with $\mu(B) > 0$ we have that for almost every $x \in B$, for every compact $K \subset M$, there is a $g \in M \setminus K$ so that $gx \in B$, in other words, if Poincaré recurrence holds for this action. This certainly seems the very least one needs to assume about a measurable action to get nontrivial dynamics.

In the very special case of \mathbb{Z} - or \mathbb{R} -actions and μ quasi-invariant (i.e., while μ may not be conserved by the action, the measure class is), what we call recurrence is known as conservativity and plays an important role; for example, see [Aar97, Sect. 1.1].

³ The restriction to $\mathrm{SL}(n, \mathbb{Z})$ is not a technicality: there are lattices, even cocompact lattices, which have non-algebraic H -invariant measures with positive entropy with respect to a one-parameter subgroup of H (indeed, the Rees examples alluded to earlier can have positive entropy). Note also that even for $\mathrm{SL}(n, \mathbb{Z})$, under the weaker assumptions of Theorem 3.4, μ need not be the G -invariant measure on $\Gamma \backslash G$, but can be an algebraic measure on a proper subspace.

We now return for a moment to \mathbb{T} . B. Host gave a proof of Rudolph's theorem regarding invariant measures on \mathbb{T} along the following lines: first he showed that every \times_n -invariant measure μ all of whose \times_n -ergodic components have positive entropy are in an appropriate sense (which will be defined momentarily) recurrent under the action of the **additive** group

$$N_n = \left\{ \frac{q}{n^k} : k \in \mathbb{N}, q \in \mathbb{Z}/(n^k)\mathbb{Z} \right\}.$$

Host then shows that while there is an abundance of measures on \mathbb{T} invariant under \times_m , if m and n are relatively prime the only \times_m -invariant measure which is N_n -recurrent is Lebesgue measure.

In fact, Host proved more: if one assumes that μ is \times_n -invariant, and the entropy of the \times_n -ergodic components of μ are bounded from below by some positive number, then μ satisfies a stronger quantitative condition that implies N_n -recurrence, and that for any measure μ which satisfies this condition, the orbit of μ -almost every x under successive iterates of \times_m is uniformly distributed with respect to the Lebesgue measure on \mathbb{T} (which, by the pointwise ergodic theorem, clearly implies that if μ is also \times_m -invariant then μ is Lebesgue measure). Extensions of Host's result to the multidimensional case can be found in [Hoss00, Mei98]; other related papers that in particular treat the case of m, n which are not relatively prime can be found in [LMP99, Lin01b].

It is customary in the literature to refer to a space X on which a group G acts as a G -space (to avoid pathologies, it is helpful to assume, e.g., that G acts freely on a dense subset of X). A generalization of this is the notion of a (G, T) -space, for a nice⁴ group G , and T a homogeneous space for G (i.e., a locally compact metric space T on which G acts transitively) with a distinguished point $e \in T$.

Definition 4.1: A locally compact metric space X is said to be a (G, T) -space if there is some open cover \mathfrak{T} of X by relatively compact sets, and for every $U \in \mathfrak{T}$ and $x \in U$ a map $t_{U,x}: T \rightarrow X$ with the following properties:

- (A-1) For every $x \in U \in \mathfrak{T}$, we have that $t_{U,x}(e) = x$.
- (A-2) The map $(x, t) \mapsto t_{U,x}(t)$ is a continuous map $U \times T \rightarrow X$.
- (A-3) For any $x \in U \in \mathfrak{T}$, for any y in the image $\text{Image } t_{U,x}$ of $t_{U,x}$ and $V \in \mathfrak{T}$ containing y , there is a $\theta \in G$ so that

$$(4.1) \quad t_{V,y} \circ \theta = t_{U,x}.$$

In particular, for any x, y, U, V as above, $\text{Image } t_{U,x} = \text{Image } t_{V,y}$ and $t_{U,x}$ is injective if and only if $t_{V,y}$ is.

⁴ Locally compact separable and metrizable.

- (A-4) For any $U \in \mathfrak{T}$ there is some open neighborhood B^U of the distinguished point $e \in T$ so that $t_{U,x}$ is injective on $\overline{B^U}$ for all $x \in U$.
- (A-5) For a dense set of $x \in X$, for some (equivalently for every) $U \in \mathfrak{T}$ containing X the map $t_{U,x}$ is injective.

We will be only interested in the case where G acts by isometries, in other words when G is a subgroup of the group $\text{Isom}(T)$ of isometries of T . For simplicity, we shall call a $(\text{Isom}(T), T)$ -space simply a T -space.

In less technical terms, a T -space is a space which is foliated in a continuous way by leaves, such that on every leaf there is a well-defined metric, and every leaf is locally isometric to T with respect to this metric. Furthermore, for most leaves this is actually an isometry between the leaf and T . A (G, T) -space for G a proper subgroup of $\text{Isom}(T)$ is a special kind of T -space with some additional structure.

Definition 4.2: A measure μ on any T -space X is said to be **T -recurrent** if for every $B \subset X$ with $\mu(B) > 0$, for μ -almost every $x \in B$ (and some $U \in \mathfrak{T}$ containing x) for every compact $K \subset T$,

$$t_{U,x}(T \setminus K) \cap B \neq \emptyset.$$

If μ is a Radon⁵ measure on a (G, T) -space, it induces on every T -leaf a σ -finite measure which is well defined up to a multiplicative scalar. We can either think of these measures as σ -finite but no longer Radon measures on X , or use the isomorphism between typical T -leaves and T and consider these measures as nice Radon measures on T . We adopt here the latter point of view, and so formally these conditional measures are a system $\{\mu_{T,x}^U\}_{U \in \mathfrak{T}, x \in U}$ of Radon measures on T so that for every U , the map $x \mapsto \mu_{T,x}^U$ is Borel measurable⁶ and so that there is a set of full measure $X' \subset X$ so that for every $x, y \in X'$ and $U, V \in \mathfrak{T}, g \in G$ such that $t_{U,x} \circ g = t_{V,y}$ then

$$\mu_{T,x}^U = g_* \mu_{T,y}^V,$$

i.e., so that formally $(t_{U,x})_* \mu_{T,x}^U$ is equal to $(t_{V,y})_* \mu_{T,y}^V$. Defining these measures carefully is slightly tricky (see [Lin03a, Sect. 3]; [KK01, Lin03b] may also be useful).

These conditional measures reflect the dynamics of μ with respect to the foliation: for example, if T is a group and the T -structure comes from some

⁵ I.e., a regular Borel measure which is finite on compact sets.

⁶ With respect to the weak* topology on the space of Radon measures on T as the dual to the space of compactly supported continuous functions.

action of T , then μ is T -invariant if and only if $\mu_{T,x}$ is Haar measure for almost every x (the dependence on the set U in the atlas \mathfrak{T} is trivial and therefore omitted in this case). Furthermore, and this is an important fact, if μ is a probability measure on a (G, T) -space then μ is T -recurrent if and only if $\mu_{T,x}^U$ is an infinite measure a.s.⁷

Jointly with Klaus Schmidt, and using a method reminiscent of (but distinct from) that of Host, we have proved

THEOREM 4.3 (Schmidt and Lindenstrauss [LS03]): *Let α be a nonexpansive, ergodic and totally irreducible automorphism of a compact connected Abelian group X (for example, let α be a nonhyperbolic toral automorphism); and let μ be an α -invariant probability measure on X . Then μ is recurrent with respect to the central foliation of X if and only if it is Lebesgue measure on X .*

This theorem also implies that if, for example, μ is weakly mixing for α then there is a Borel subset of X , which intersects every central leaf in at most one point, and which has full μ -measure. A similar, but slightly weaker statement holds for a general invariant measure which is singular with respect to Lebesgue measure. Thus, in some sense the only invariant measure on X which is truly nonhyperbolic is Lebesgue measure.

We contrast this with the situation of another partially hyperbolic dynamical system: the time one map g_1 on a negatively curved compact manifold, where **all** g_1 -invariant measures are recurrent with respect to the central foliation, which in this case is simply the foliation of the unit tangent bundle of the manifold into orbits of the geodesic flow.

5. Rigidity of Hecke recurrent measures on arithmetic surfaces

A similar theorem can be proved in the context of homogeneous spaces, and can be used to settle an important special case of a conjecture of Rudnick and Sarnak [RS94] regarding the behavior of eigenfunctions of the Laplacian on negatively curved manifolds.

Take $X = \Gamma \backslash \mathrm{PGL}(2, \mathbb{R})$, where Γ is either a congruence subgroup of $\mathrm{PGL}(2, \mathbb{Z})$ or of certain lattices that arise from quaternionic division algebras over \mathbb{Q} that are unramified over \mathbb{R} . The latter lattices are slightly harder to define but have the advantage that X is compact.

In both cases, for all but finitely many prime p , there is a map T_p from X to $(p+1)$ -tuples of points of X called the **Hecke correspondence**. For example,

⁷ For \mathbb{Z} -actions which preserve the measure class of μ , this result is known as the Halmos Recurrence Theorem (see [Aar97, Sect. 1.1]).

if $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, one can define T_p of a point $x \in X$ as follows: choose some $g \in \mathrm{SL}(2, \mathbb{R})$ so that $x = \Gamma g$. Then

$$T_p(x) = \left\{ \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} g, \Gamma \begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix} g, \dots, \Gamma \begin{pmatrix} p & 0 \\ p-1 & 1 \end{pmatrix} g, \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} g \right\}.$$

Notice that while each individual element of the right-hand side depends on the choice of g , the set of $p+1$ points depends only on x . The above formula also shows that for any $x \in X, g \in \mathrm{PGL}(2, \mathbb{R})$,

$$T_p(xg) = T_p(x)g.$$

Using the Hecke correspondence we can define operators called the **Hecke operators** on $L^2(X)$ by

$$(5.1) \quad T_p(f)[x] = \sum_{y \in T_p(x)} f(y),$$

which play a very important role in the spectral theory of X . The Hecke correspondences have some additional nice properties, which are not entirely obvious from the way we defined them here. For example, if $y \in T_p(x)$ then $x \in T_p(y)$, which means that the Hecke operators are self-adjoint. Furthermore, for any p and q , the operators T_p and T_q commute.

Fix a prime p . We define the following equivalence relation on X : $x \sim y$ if there is a chain of points x_i with $x_0 = x$, $x_n = y$ and $x_{i+1} \in T_p(x_i)$. We can give each equivalence class a structure of a $(p+1)$ -regular graph if we connect every x with every point in $T_p(x)$, and this graph turns out to be a $(p+1)$ -regular tree for all x . By (5.1) this equivalence relation, and indeed even the structure of every equivalence class as a $(p+1)$ -regular tree, is invariant under left translation by elements of $\mathrm{PGL}(2, \mathbb{R})$. It is not hard to see that if T is a $(p+1)$ -regular tree, this partition of X into $(p+1)$ -regular trees is a T -space.

Consider now the action of the one-parameter group

$$A = \left\{ a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}$$

on X . This action is equivalent to the geodesic flow on the hyperbolic manifold $\Gamma \backslash \mathbb{H}$ with \mathbb{H} the hyperbolic plane, which is a prototypical example where measure rigidity does not hold. However, it seems plausible that there will not be many measures μ which are invariant under A and are T -recurrent. At the current state of our knowledge, this still seems unapproachable. But if one assumes in addition an entropy condition one can obtain the following:

THEOREM 5.1 (Lindenstruass [Lin03a]): *Suppose μ is an A -invariant and T -recurrent measure on X , where T is a $(p+1)$ -regular tree as above. Suppose that μ -almost every A ergodic component of μ has positive entropy. Then μ is Lebesgue measure on X .*

We remark that, as in Rudolph's theorem, a substantial source of difficulty is that μ need not be ergodic under A .

Somewhat surprisingly, the main actor in the proof of Theorem 5.1 is not the action of the flow A which preserves the measure, but rather a different flow: the horocycle flow given by the (right) action of the group

$$U^+ = \{u(s) : s \in \mathbb{R}\}, \quad u(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

on X (which a priori need not preserve either μ or even the measure class of μ). Note that since A normalizes the group U^+ , the orbits of the group U^+ are preserved under $a(t)$ and are uniformly contracted by this flow, a relation which links these two actions in many subtle ways.

Indeed, the proof of Theorem 5.1 has many similarities to Ratner's earlier work on rigidity of unipotent flows [R82b, R82a, R83], and in particular uses Ratner's **H-property** of horocycle flows (a related, but distinct property, the **R-property**, lies at the heart of Ratner's subsequent proof of the Raghunathan Conjecture; see [R92] for more details). This property is a quantitative form of the following qualitative phenomenon: consider the trajectories under U^+ of two nearby points $x, x' \in X$. Then there are two possibilities⁸:

- (H-1) $xu(s), x'u(s)$ remain close for all s , which can occur only if $x' = xu(\eta)$ for some small $\eta \in \mathbb{R}$.
- (H-2) Otherwise, there will be some smallest $s > 0$ where $xu(s), x'u(s)$ differ by a noticeable amount (say are of distance δ from each other, for some fixed but small δ). Then most of the distance between $xu(s)$ and $x'u(s)$ is along the direction of the U^+ flow, i.e., there is some s' (roughly of order δ) so that $xu(s)$ is close to $x'u(s+s')$ (see Figure 1).

The proof of the H-property is not difficult: it is merely an exercise in multi-

⁸ For simplicity, we assume X compact; minor modifications are needed for the nonuniform case

plying 2×2 matrices, but it has many far-reaching implications.

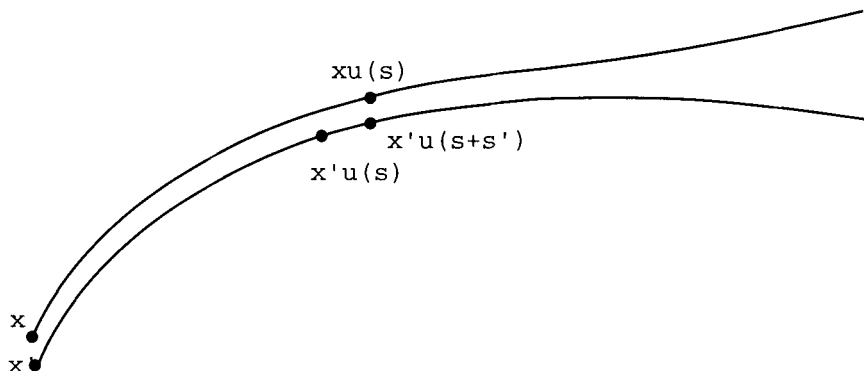


Figure 1. Ratner's H-property.

We now give a rough sketch of the proof of Theorem 5.1, explaining how the H-property of the horocycle flow U^+ can be used to study measures invariant under the A -flow.

It follows from a slight generalization of a lemma of Einsiedler and Katok and a Fubini argument that if μ is $a(t)$ -invariant, there is some set of full measure $X' \subset X$ so that every $x, y \in X'$ which are in the same Hecke tree (i.e., $x \sim y$) satisfy that $\mu_{x,U^+} = \mu_{y,U^+}$. Using Hecke recurrence starting from almost any $x \in X'$ we can find a $y \sim x$ in X' which is arbitrarily close to x .

By Luzin's theorem the map $z \mapsto \mu_{z,U^+}$ is continuous in the weak* topology on a set of large measure, so since y is nearby x we can expect that μ_{x,U^+} is fairly similar to μ_{y,U^+} . Of course this is no big surprise, and there is no contradiction so far since we know that in fact μ_{x,U^+} is actually equal to μ_{y,U^+} . But now consider $yu(s)$ and $xu(s)$ which are still Hecke equivalent, for some s large enough so that (H-2) above holds. Then by the properties of conditional measures typically $\mu_{xu(s),U^+} = \mu_{yu(s),U^+}$. But now $xu(s)$ and $yu(s)$ are no longer close: indeed, $xu(s)$ is much closer to $yu(s+s')$ for some macroscopic s' . Thus we would expect $\mu_{xu(s),U^+}$ to be fairly similar to $\mu_{yu(s+s'),U^+} = \mu_{xu(s+s'),U^+}$. This hints we may be able to show that μ is U^+ -invariant. Similarly, one shows that μ is invariant under

$$U^- = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \right\},$$

so μ is Haar measure.

There are many tricky issues in making this crude idea work. One ingredient is a maximal inequality (sometimes known as a maximal ergodic theorem) which appears in [Lin03a, Appendix] and is joint with D. Rudolph. This ergodic theorem generalizes a maximal inequality of Hurewicz [Hur44]. A similar idea has been used in Rudolph's proof of Bernoullicity of the Patterson–Sullivan measure [Rud82].

6. An application to quantum unique ergodicity

In this section we finally describe the quantum unique ergodicity conjecture of Rudnick and Sarnak, and explain how it is related to Theorem 5.1.

CONJECTURE 7 (Rudnick and Sarnak [RS94]): *Let M be a compact hyperbolic surface, and let ϕ_i be a sequence of linearly independent eigenfunctions of the Laplacian Δ on M , normalized to have L^2 -norm one. Then the probability measures $\tilde{\mu}_i$ defined by $\tilde{\mu}_i(A) = \int_A |\phi_i(x)|^2 d\mathcal{H}_M(x)$ tend in the weak* topology to the uniform measure \mathcal{H}_M .*

A. I. Šnirel'man, Y. Colin de Verdière and S. Zelditch have shown in great generality (specifically, for any manifold on which the geodesic flow is ergodic) that if one omits a subsequence of density 0 the remaining $\tilde{\mu}_i$ do indeed converge to \mathcal{H}_M [Šni74, CdV85, Zel87]. An important component of their proof is the **microlocal lift** of any weak* limit $\tilde{\mu}$ of a subsequence of the $\tilde{\mu}_i$. The microlocal lift of $\tilde{\mu}$ is a measure μ on the unit tangent bundle SM of M whose projection to M is $\tilde{\mu}$, and most importantly it is always invariant under the geodesic flow on SM . We shall call any measure μ on SM arising as a microlocal lift of a weak* limit of $\tilde{\mu}_i$ a **quantum limit**. Thus a slightly stronger form of Conjecture 7 is the following conjecture, also due to Rudnick and Sarnak:

CONJECTURE 8 (Quantum Unique Ergodicity Conjecture): *For any compact hyperbolic surface M , the only quantum limit is the uniform measure \mathcal{H}_{SM} on SM .*

Consider now $M = \Gamma \backslash \mathbb{H}$, for Γ one of the arithmetic lattices considered above. If Γ is a lattice arising from a quaternionic division algebra over \mathbb{Q} , then M is a compact hyperbolic surface, precisely the kind of surface considered in Conjectures 7 and 8.

If Γ is a congruence sublattice of $\mathrm{PGL}(2, \mathbb{Z})$ then M has finite volume, but is not compact. It is not hard to modify Conjectures 7 and 8 so that they cover also this case. A special property of these very special surfaces of finite

volume is that their spectral theory is not too far from that of compact surfaces of the same area: there is continuous spectrum (which of course does not exist for compact surfaces), but it is very well understood, and as shown by Selberg [Sel56] the discrete spectrum of the Laplacian on these noncompact surfaces obeys Weyl's law for compact surfaces of the same area. Since Luo, Sarnak and Jakobson proved equidistribution for the continuous spectrum [LS95, Jak97], even for these noncompact surfaces one only needs to study the weak* limits of measures which arise from L^2 -normalized eigenfunctions of the Laplacian.

When looking at eigenfunctions of the Laplacian on the arithmetic surfaces, it is natural to consider joint eigenfunctions of both the Laplacian and of all Hecke operators. Since the Hecke operators are self-adjoint operators that commute with each other and with the Laplacian, one can always find an orthonormal basis of the subspace of $L^2(M)$ which corresponds to the discrete part of the spectrum of the Laplacian consisting of such joint eigenfunctions. Furthermore, if the spectrum is simple, eigenfunctions of the Laplacian are automatically eigenfunctions of all Hecke operators.

These joint eigenfunctions of the Laplacian and all Hecke operators, usually referred to as Hecke–Maass cusp forms, are of great interest in number theory. Much more is known about them than about arbitrary eigenfunctions of the Laplacian on a hyperbolic surface, and so considerable interest has been centered on the following special case of Rudnick and Sarnak's quantum unique ergodicity conjecture:

QUESTION 6.1 (Arithmetic Quantum Unique Ergodicity): *Let M be $\Gamma \backslash \mathbb{H}$ for Γ a congruence as above, and let ϕ_i be a sequence of linearly independent joint eigenfunctions of the Laplacian Δ and all Hecke operators on M , normalized to have L^2 -norm one. What are the possible microlocal lifts of weak* limits of the probability measures $\tilde{\mu}_i$ defined as above by $\tilde{\mu}_i(A) = \int_A |\phi_i(x)| d\mathcal{H}_M(x)$ (we shall refer to these lifts as arithmetic quantum limits)?*

We review what was known previously regarding this question. T. Watson [Wat01] has shown that assuming GRH, the only possible arithmetic quantum limit is the uniform measure on $\Gamma \backslash \mathrm{PGL}(2, \mathbb{R})$. In fact, GRH gives an optimal rate of convergence.

Unconditionally, Sarnak has been able to show that a certain subsequence of the Hecke–Maass forms, the CM-forms, satisfies this conjecture [Sar01].

Also unconditionally, Rudnick and Sarnak [RS94] showed that arithmetic quantum limits cannot be supported on finitely many closed geodesics; similar ideas were used by the author and Wolpert [Lin01a, Wol01]. A small variation

of [Lin01a] gives that any arithmetic quantum limit is Hecke recurrent for every prime p . Refining these methods, jointly with J. Bourgain, we have been able to prove the following:

THEOREM 6.2 (Bourgain and Lindenstrauss [BL03]): *Every ergodic component of an arithmetic quantum limit has entropy $> \frac{2}{9}$ (in this normalization, the entropy of the uniform Lebesgue measure is 2).*

Using Theorem 6.2 together with the Hecke recurrence, we can now deduce from Theorem 5.1 the following theorem regarding arithmetic quantum unique ergodicity:

THEOREM 6.3 (Arithmetic Quantum Unique Ergodicity): *Let M be $\Gamma \backslash \mathbb{H}$ for Γ a congruence lattice as above. Then any arithmetic quantum limit for M is $c\mathcal{H}_M$ for some $c \in [0, 1]$; in the compact case, $c = 1$.*

This (almost) proves the arithmetic case of the quantum unique ergodicity conjecture of Rudnick and Sarnak (the only missing piece is showing $c = 1$ also in the non-compact case).

Using the same general strategy, L. Silberman and A. Venkatesh have been able to prove arithmetic quantum unique ergodicity for other $\Gamma \backslash G/K$, specifically for locally symmetric spaces arising from division algebras of prime degree. While the strategy remains the same, several new ideas are needed for this extension, in particular a new micro-local lift for higher rank groups [SV04].

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